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AN EQUIVALENT LINEAR PROGRAMMING PROBLEM

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SUMMARY

An equivalent formulation for a standard linear programming problem is developed. For the case where the number of variables is twice the number of equations m, the equivalent problem has the same size but has the inverses of the first and second m columns of the matrix of coefficients.

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Any linear program can be converted to an equivalent linear program by

(a) dualizing

(b) replacing a system of equations by an equivalent system in the same variables

(c) elimination of variables unrestricted in sign

(d) elimination of variables restricted in sign by the Fourier-Motzkin Elimination Method.

Our purpose is to add to the collection by proving an interesting relation for the case of 2m nonnegative variables in m equations which permits substituting another system of the same size by replacing the first m columns by its transpose inverse and the second m columns by the negative of its transpose inverse. A generalization of this result for the case of m equations in n nonnegative variables will also be developed.

The theorem was observed recently as part of the authors' investigations of partitioning methods. It is not known at this early date whether any significant applications can be made of it but this may well prove to be the case since equivalences have proved to be powerful tools in the past.

The standard L.P. problem is to find $x_j \geq 0$ and minimum z satisfying

$$(1)\qquad \begin{array}{l} a_{11}x_1 + \cdots + a_{1m}x_m + a_{1,m+1}x_{m+1} + \cdots + a_{1,2m}x_{2m} = b_1 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mm}x_m + a_{m,m+1}x_{m+1} + \cdots + a_{m,2m}x_{2m} = b_m \\ c_1\,x_1 + \cdots + c_m\,x_m + c_{m+1}\,x_{m+1} + \cdots + c_{2m}\,x_{2m} = z \quad (\mathrm{Min})\ . \end{array}$$

where $x_j \geq 0$. We may write this in matrix form

$$(2)\qquad \begin{array}{ll} B_1X_1 + B_2X_2 = q & \qquad X_1 \geq 0,\ X_2 \geq 0 \\ \gamma_1X_1 + \gamma_2X_2 = z \quad (\mathrm{Min}) & \end{array}$$

where

$$(3)\qquad \begin{array}{l} X_1 = (x_1,\ x_2\ ,\ldots,x_m)^T \qquad (T = \text{transpose}) \\ X_2 = (x_{m+1},\ x_{m+2},\ldots,x_{2m})^T \\ q = (b_1,\ b_2\ ,\ldots,b_m)^T \\ \gamma_1 = (c_1,\ c_2\ ,\ldots,c_m) \\ \gamma_2 = (c_{m+1},\ c_{m+2},\ldots,c_{2m})\ . \end{array}$$

THEOREM: <u>If</u> B_1^{-1} <u>and</u> B_2^{-1} <u>exist, then the linear programming problem</u>

$$(4)\qquad \begin{array}{l} (B_1^{-1})^T\,\bar{X}_1 - (B_2^{-1})^T\,\bar{X}_2 = (\bar{\gamma}_1 - \bar{\gamma}_2)^T \\ \qquad - \bar{q}_2^T\,\bar{X}_2 = v(\mathrm{Max}) \end{array}$$

<u>where</u> $\bar{X}_1 \geq 0$, $\bar{X}_2 \geq 0$ <u>and</u>

$$(5)\qquad B_1^{-1}\,q = \bar{q}_1,\ B_2^{-1}\,q = \bar{q}_2,\ \gamma_1B_1^{-1} = \bar{\gamma}_1,\ \gamma_2B_2^{-1} = \bar{\gamma}_2$$

<u>is equivalent to the dual of (1) or (2).</u>

<u>Proof</u>: Consider the system:

$$(6)\qquad B_1X_1 = Y \qquad B_2X_2 + Y = q \qquad (X_1 \geq 0,\ X_2 \geq 0)$$
$$\gamma_1X_1 + \gamma_2X_2 = z \quad (\text{Min})$$

where $Y = (y_1, y_2, \ldots, y_m)$ are an auxilliary set of variables unrestricted in sign.

We may rewrite this

$$(7)\qquad X_1 = B_1^{-1}Y,\ X_2 = B_2^{-1}q - B_2^{-1}Y \qquad X_1 \geq 0,\ X_2 \geq 0$$
$$\bar{\gamma}_1Y - \bar{\gamma}_2Y = z - \gamma_2\bar{q}_2 \quad (\text{Min})$$

or

$$(8)\qquad B_1^{-1}Y \geq 0$$
$$-B_2^{-1}Y \geq -\bar{q}_2$$
$$(\bar{\gamma}_1 - \bar{\gamma}_2)Y = z' \quad (\text{Min}) \qquad z' = z - \gamma_2\bar{q}_2\ .$$

Noting that Y is unrestricted in sign it is easy to see that the dual of (8) is (4). Hence elements of vector Y are the simplex multipliers associated with the optimal solution of (4) and their substitution into (7) yields the required optimal solution to (2).

For example if

$$[a_{ij}] = \left[\begin{array}{cccc|cccc} 1 & & & & 1 & 1 & 1 & 1 \\ 1 & 1 & & & & 1 & 1 & 1 \\ 1 & 1 & 1 & & & & 1 & 1 \\ 1 & 1 & 1 & 1 & & & & 1 \end{array}\right]$$

the problem equivalent to the dual would have coefficients

$$\left[\begin{array}{cccc|cccc} 1 & & & & 1 & -1 & & \\ & -1 & & & & 1 & -1 & \\ & -1 & 1 & & & & 1 & -1 \\ & & -1 & 1 & & & & 1 \end{array}\right]$$

which has less non-zero entries and incidently shows that the original problem was a camouflaged transportation problem [which has been observed by Fulkerson and others in a somewhat more general context.]

2. In the case $n = km$ one can obtain analogous results. We illustrate this for an example where $k = 4$.

First it is assumed that one can permute the variables so as to form four groups of m variables each whose corresponding column vectors form matrices B_i, $i = 1,\dots 4$ such that B_i^{-1} exist. Then using analogous notation to that of §1, the original problem can be written as

(9) $$\sum_1^4 B_i X_i = q \qquad X_i \geq 0, \quad i = 1,2,3,4$$

$$\sum_1^4 \gamma_i X_i = z \text{ (Min)}.$$

Define column vectors Y_i, $(i = 1,2,3)$, whose elements are unrestricted in sign by the first three relations of (10), then the last relation is implied by (9) and conversely.

(10) $B_1X_1 = Y_1,\ B_2X_2 + Y_1 = Y_2,\ B_3X_3 + Y_2 = Y_3,\ B_4X_4 + Y_3 = q$

Since $X_1 = B_1^{-1}\, Y_1,\ X_2 = B_2^{-1} Y_2 - B_2^{-1}\, Y_1,\ X_3 = B_3^{-1}\, Y_3 - B_3^{-1}\, Y_2$

and $X_4 = B_4^{-1}\, q - B_4^{-1}\, Y_3$ we can write (9) as

(11)
$$\begin{aligned}
B_1^{-1}\, Y_1 & & & \geq 0 \\
-B_2^{-1}\, Y_1 + B_2^{-1}\, Y_2 & & & \geq 0 \\
& - B_3^{-1}\, Y_2 + B_3^{-1}\, Y_3 & & \geq 0 \\
& & - B_4^{-1} Y_{33} & \geq - B_4^{-1}\, q = \bar{q}_4 \\
(\bar{\gamma}_1 - \bar{\gamma}_2)Y_1 + (\bar{\gamma}_2 - \bar{\gamma}_3)Y_2 + (\gamma_3 - \gamma_4)Y_3 & = z' \text{ (Min)}
\end{aligned}$$

and as before the dual of (11), setting $A_i = [B_i^{-1}]^T$, is

(12)
$$\begin{aligned}
& & \bar{X}_i & \geq 0 \\
A_1\bar{X}_1 - A_2\bar{X}_2 & & & = \bar{\gamma}_1 - \bar{\gamma}_2 \\
& A_2\bar{X}_2 - A_3\bar{X}_3 & & = \bar{\gamma}_2 - \bar{\gamma}_3 \\
& & A_3\bar{X}_3 - A_4\bar{X}_4 & = \bar{\gamma}_3 - \bar{\gamma}_4 \\
-A_1\bar{X}_1 & & + A_4\bar{X}_4 & = \bar{\gamma}_4 - \bar{\gamma}_1 \\
& & - \bar{q}_4^T X_4 & = v\text{(Max)}.
\end{aligned}$$

In (12) the redundant fourth vector equation equal to the negative sum of the first three equations is included to

bring out the symmetry of the form of the problem and to facilitate generating the other equivalent systems by adding relations.

3. In case n is not a multiple of m, one can add dummy column vectors to fill out an m by m matrix so that its inverse exists, while adjusting the minimizing form with coefficients sufficiently large to drive out the corresponding **dummy variables** from consideration in the optimal solution.

To illustrate, if $n = 2m - 1$, add an extra vector to fill out B_2 with $c_{2m} = M$ large enough so that x_{2m} vanishes in an optimum solution.

(13) $$B_1 X_1 + B_2 X_2 = q, \qquad X_1 \geq 0$$

$$\sum_1^{2m-1} c_i x_i + M x_{2m} = z(\text{Min})$$

which, as before, is equivalent to

(14) $$A_1 \bar{X}_1 - A_2 \bar{X}_2 = \bar{\gamma}_1 - \bar{\gamma}_2, \qquad \bar{X}_1 \geq 0$$

$$-\bar{q}_2^T \bar{X}_2 = v(\text{Max}).$$